Sequences

A **Sequence** is a list of numbers written in order.

$$\{a_1,\ a_2,\ a_3,\ \dots\}$$

The sequence may be infinite. The n **th term** of the sequence is the n th number on the list. On the list above

$$a_1 = \text{1st term},\ \ a_2 = \text{2 nd term},\ \ a_3 = \text{3 rd term},\ \ \text{etc....}$$

- **Example** In the sequence $\{1, 2, 3, 4, 5, 6, \dots\}$, we have $a_1 = 1,\ \ a_2 = 2,\ \ \dots.$ The n^{th} term is given by $a_n = n$.

Some sequences have **patterns**, some do not.

- **Example** If I roll a 20 sided die repeatedly, I generate a sequence of numbers, which have no pattern.
- **Example** The sequences

  $$\{1, 2, 3, 4, 5, 6, \dots\}$$

  and

  $$\{1,\ -1,\ 1,\ -1,\ 1,\ \dots\}$$

  have patterns.

Formula for a_n

Sometimes we can give a **formula for the n th term of a sequence, $a_n = f(n)$.**
Example For the sequence $\{1, 2, 3, 4, 5, 6, \dots\}$, we can give a formula for the n th term. $a_n = n$.
Example Assuming the following sequences follow the pattern shown, give a formula for the n-th term:

- $\{1,\ -1,\ 1,\ -1,\ 1,\ \dots\}$
- nth term $= a_n = (-1)^{n+1}$.
- $\{-1/2,\ 1/3,\ -1/4,\ 1/5,\ -1/6,\ \dots\}$
- nth term $= a_n = \frac{(-1)^n}{n+1}$.

Factorials are commonly used in sequences

$$0! = 1,\ \ 1! = 1, \quad 2! = 2\cdot 1, \quad 3! = 3\cdot 2\cdot 1, \quad \dots, \quad n! = n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 1.$$

Example Find a formula for the n th term in the following sequence

$$\left\{\frac{2}{1}, \frac{4}{2}, \frac{8}{6}, \frac{16}{24}, \frac{32}{120}, \dots \quad , a_n = \qquad , \right\}$$

- nth term $= a_n = \frac{2^n}{n!}$.

Different ways to represent a sequence

Below we show **3 different ways to represent the sequences given**:

A.

$$\left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \dots,\ \frac{n}{n+1},\ \dots\right\}, \qquad \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}, \qquad a_n = \frac{n}{n+1}.$$

B.

$$\left\{\frac{-3}{3},\ \frac{5}{9},\ \frac{-7}{27},\ \dots,\ (-1)^n\frac{(2n+1)}{3^n},\ \dots\right\},$$

$$\left\{(-1)^n\frac{(2n+1)}{3^n}\right\}_{n=1}^{\infty}, \qquad a_n = (-1)^n\frac{(2n+1)}{3^n}.$$

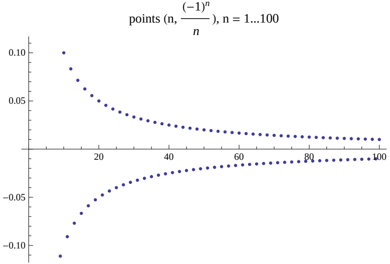
C.

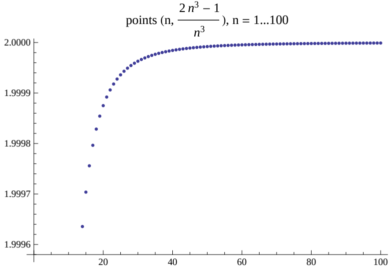
$$\left\{\frac{e}{1},\ \frac{e^2}{2},\ \frac{e^3}{6},\ \dots,\ \frac{e^n}{n!},\ \dots\right\}, \qquad \left\{\frac{e^n}{n!}\right\}_{n=1}^{\infty}, \qquad a_n = \frac{e^n}{n!}.$$

Graph of a sequence

A sequence is a function from the positive integers to the real numbers, with $f(n) = a_n$. We can draw a graph of this function as a set of points in the plane. The points on the graph are $(1, a_1)$, $(2, a_2)$, $(3, a_3)$, $\ldots$ $, (n, a_n)$, $\ldots$

Example Graph the sequences $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ and $\left\{\frac{2n^3-1}{n^3}\right\}_{n=1}^{\infty}$.





We can see from these pictures that the graphs get closer to a horizontal asymptote as $n \to \infty$, $y = 0$ on the left and $y = 2$ on the right. Algebraically this means that as $n \to \infty$, we have $\frac{(-1)^n}{n} \to 0$ and $\frac{2n^3-1}{n^3} \to 2$.

Limit of a sequence

Definition A sequence $\{a_n\}$ has **limit** L if we can make the terms a_n as close as we like to L by taking n sufficiently large. We denote this by

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \ \text{ as } \ n \to \infty.$$

If $\lim_{n\to\infty} a_n$ exists (is finite), we say the sequence **converges** or is convergent. Otherwise, we say the sequence **diverges**.

Graphically: If $\lim_{n\to\infty} a_n = L$, the graph of the sequence $\{a_n\}_{n=1}^{\infty}$ has a unique horizontal asymptote $y = L$.

Equivalent Definition A sequence $\{a_n\}$ has limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \ \text{ as } \ n \to \infty$$

if for every $\epsilon > 0$ there is and integer N with the property that

$$\text{if} \ \ n > N \quad \text{then} \quad |a_n - L| < \epsilon.$$

Determining if a sequence is convergent.

Using our previous knowledge of limits :

Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, where n is an integer, then $\lim_{n\to\infty} a_n = L$.

Example Determine if the following sequences converge or diverge:

$$A. \ \left\{\frac{2^n - 1}{2^n}\right\}_{n=1}^{\infty}, \qquad B. \ \left\{\frac{2n^3 - 1}{n^3}\right\}_{n=1}^{\infty}$$

- A. $\lim_{x\to\infty} \frac{2^x - 1}{2^x} = \lim_{x\to\infty} \frac{1 - 2^{-x}}{1} = 1.$
- Therefore the sequence $\left\{\frac{2^n-1}{2^n}\right\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n-1}{2^n} = 1.$
- B. $\lim_{x\to\infty} \frac{2x^3 - 1}{x^3} = \lim_{x\to\infty} \frac{2 - 1/x^3}{1} = 2.$
- Therefore the sequence $\left\{\frac{2n^3-1}{n^3}\right\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^3-1}{n^3} = 2.$

L'Hospital's rule

We can use L'Hospital's rule to determine the limit of $f(x)$ if we have an indeterminate form.

Example Is the following sequence convergent?

$$\left\{\frac{n}{2^n}\right\}_{n=1}^{\infty}$$

- $\lim_{x\to\infty} \frac{x}{2^x} = $ (by l'Hospital) $\lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0$.
- Therefore the sequence converges and $im_{n\to\infty} \frac{n}{2^n} = 0$.

Diverging to ∞. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M, there is an integer N with the property

$$\text{if} \quad n > N, \qquad \text{then} \qquad a_n > M.$$

In this case we say the sequence $\{a_n\}$ **diverges to infinity**.

Note: If $\lim_{x\to\infty} f(x) = \infty$ and $f(n) = a_n$, where n is an integer, then $\lim_{n\to\infty} a_n = \infty$.

Important sequence/limit

Example Show that the sequence $\{r^n\}_{n=1}^{\infty}$, $r \geq 0$, converges if $0 \leq r \leq 1$ and diverges to infinity if $r > 1$.

- $\lim_{n\to\infty} r^n = \lim_{x\to\infty} r^x = \lim_{x\to\infty} e^{x \ln r}$.
- $\lim_{x\to\infty} e^{x \ln r} = \begin{cases} 0 & \text{if } r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$
- Therefore the sequence $\{r^n\}_{n=1}^{\infty}$, $r \geq 0$, converges if $0 \leq r \leq 1$ and diverges to infinity if $r > 1$.

Rules of Limits

The usual **Rules of Limits** apply:

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is any constant then

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } \; p > 0 \; \text{ and } \; a_n > 0$$

In fact if $\lim_{n\to\infty} a_n = L$ and $f(x)$ is a continuous function at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Applying the Rules of Limits

Example Determine if the following sequence converges or diverges and if it converges find the limit.

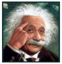
$$\left\{ \sqrt[3]{\frac{2n+1}{n}} - \frac{1}{n} \right\}_{n=1}^{\infty}.$$

- $\lim_{n\to\infty}(\sqrt[3]{\frac{2n+1}{n}} - \frac{1}{n}) = \lim_{n\to\infty} \sqrt[3]{\frac{2n+1}{n}} - \lim_{n\to\infty} \frac{1}{n}$
- $= \sqrt[3]{\lim_{n\to\infty} \frac{2n+1}{n}} - \lim_{n\to\infty} \frac{1}{n}$
- $= \sqrt[3]{\lim_{x\to\infty} \frac{2x+1}{x}} - \lim_{x\to\infty} \frac{1}{x} = \sqrt[3]{\lim_{x\to\infty} \frac{2+1/x}{1}} - 0$
- $= \sqrt[3]{2}$
- Therefore the sequence $\left\{ \sqrt[3]{\frac{2n+1}{n}} - \frac{1}{n} \right\}_{n=1}^{\infty}$ converges to $\sqrt[3]{2}$.

When there is no f(x) / Squeeze Theorem

Note We cannot always find a function $f(x)$ with $f(n) = a_n$.
The **Squeeze Theorem** or Sandwich Theorem can also be applied :

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

- **Example** Find the limit of the following sequence $\left\{\frac{2^n}{n!}\right\}_{n=1}^{\infty}$,
-  Requires a bit of cleverness, because we cannot replace $n!$ by a function $x!$.
- Certainly $\frac{2^n}{n!} > 0$ for all $n \geq 1$. So if we can find a sequence $\{c_n\}$ with $\frac{2^n}{n!} \leq c_n$ for all $n \geq 1$ and $\lim_{n\to\infty} c_n = 0$, then we can apply the squeeze theorem.
- Note that $\frac{2^n}{n!} = \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdot \dots \cdot \frac{2}{n-1} \cdot \frac{2}{n}$
- Since $\frac{2}{k} \leq 1$ if $k \geq 2$, we have $\frac{2^n}{n!} \leq 2 \cdot \frac{2}{n}$ for all $n \geq 2$.
- Since $\lim_{n\to\infty} 2 \cdot \frac{2}{n} = 0$, and $0 \leq \frac{2^n}{n!} \leq 2 \cdot \frac{2}{n}$ for all $n \geq 2$, we can conclude that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$ using the squeeze theorem.
- Therefore the sequence $\left\{\frac{2^n}{n!}\right\}_{n=1}^{\infty}$ converges to 0.

Alternating Sequences

Theorem If $\{a_n\}$ is an alternating sequence of the form $(-1)^n a'_n$ where $a'_n > 0$, then the alternating sequence converges if and only if $\lim_{n\to\infty} |a_n| = 0$ or (for the sequence described above) $\lim_{n\to\infty} a'_n \to 0$.
(also true for sequences of form $(-1)^{n+1} a'_n$ or any sequence with infinitely many positive and negative terms)

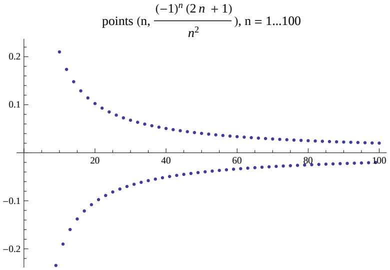
Example Determine if the following sequences converge:

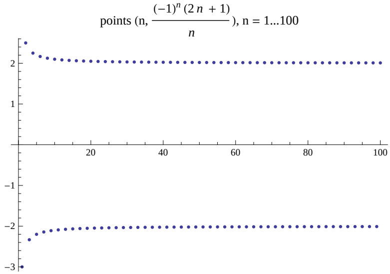
$$A. \quad \left\{(-1)^n \frac{2n+1}{n^2}\right\}_{n=1}^{\infty}, \qquad B. \quad \left\{(-1)^n \frac{2n+1}{n}\right\}_{n=1}^{\infty}$$

- A. $a_n = (-1)^n \frac{2n+1}{n^2}$.
- $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{2n+1}{n^2} = \lim_{x\to\infty} \frac{2x+1}{x^2} = \lim_{x\to\infty} \frac{(2/x)+(1/x^2)}{1} = 0$
- Therefore the sequence $\left\{(-1)^n \frac{2n+1}{n^2}\right\}_{n=1}^{\infty}$ converges to 0.
- B. $b_n = (-1)^n \frac{2n+1}{n}$.
- $\lim_{n\to\infty} |b_n| = \lim_{n\to\infty} \frac{2n+1}{n} = \lim_{x\to\infty} \frac{2x+1}{x} = \lim_{x\to\infty} \frac{2+(1/x)}{1} = 2 \neq 0$.
- Therefore the sequence $\left\{(-1)^n \frac{2n+1}{n}\right\}_{n=1}^{\infty}$ diverges.

Alternating Sequences

Geometrically, we can see the difference in the behavior of the sequences above by examining their graphs. The convergent sequence has a unique horizontal asymptote whereas the divergent sequence has two.





Monotone Bounded Sequences

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, or

$$a_1 < a_2 < a_3 < \dots.$$

A sequence $\{a_n\}$ is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$, or

$$a_1 > a_2 > a_3 > \dots.$$

A sequence $\{a_n\}$ is called **monotonic** if it is either increasing or decreasing.

Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M for which

$$a_n \leq M \quad \text{for all} \quad n \geq 1.$$

A sequence $\{a_n\}$ is **bounded below** if there is a number m for which

$$a_n \geq m \quad \text{for all} \quad n \geq 1.$$

A sequence that is bounded above and below is called **Bounded**.

Theorem Every bounded monotonic sequence is convergent.
(This theorem will be very useful later in determining if series are convergent.)

Monotone Bounded Sequences, Example

To check for monotonicity
If we have a differentiable function $f(x)$ with $f(n) = a_n$, then the sequence $\{a_n\}$ is increasing if $f'(x) > o$ and the sequence $\{a_n\}$ is decreasing if $f'(x) < o$.

Example Show that the following sequence is monotone and bounded and hence converges.

$$\{\tan^{-1}(n)\}_{n=1}^{\infty}$$

- We know that $-\frac{\pi}{2} < \tan^{-1}(n) < \frac{\pi}{2}$ for all $n > 0$.
- We also know that $\tan^{-1}(n)$ increases as n increases, since $\frac{d\tan^{-1}x}{dx} = \frac{1}{x^2+1} > 0$ for all x.
- Therefore, we can conclude that the sequence above converges.
- We could actually compute the limit here, but using the theorem for bounded monotonic sequences, we have concluded that the sequence converges without directly computing the limit.